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On properties of commutative Alexander quandles

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ABSTRACT

An Alexander quandle M_t is an abelian group M with a quandle operation $a * b = ta + (1-t)b$ where t is a group automorphism of the abelian group M. In this paper, we will study the commutativity of an Alexander quandle and introduce the relationship between Alexander quandles M_t and M_{1-t} determined by group automorphisms t and $1-t$, respectively.

Keywords: Quandle; Alexander quandle; group automorphism.

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1. Introduction

A quandle is a kind of algebraic structure that is closely related to the study of knot theory. The defining axioms of a quandle are derived from Reidemeister moves. In 1942, Takasaki introduced a *kei* which is referred to as an *involutory quandle* later [16]. In 1982, Joyce [8] and Matveev [11] independently introduced the definition of a quandle under the name "quandle" and "distributive groupoid", respectively. In their papers, they proved that if the knot quandles (the groupoids) of two knots are isomorphic, then the knots are equivalent up to orientation, that is, knot quandles can distinguish any knots up to the orientation. In this sense, a knot quandle is an almost complete invariant for knots. The classification of quandles has been studied by many researchers, see [1, 3, 9, 12, 14].

One of typical examples of quandles is an *Alexander quandle*, which is a module over $\mathbb{Z}[t, t^{-1}]$ with the operation $a * b = ta + (1-t)b$. In [4, 5], Hou gave a classification of Alexander quandles of order p^n for $n \leq 4$ and determined the (quandle) automorphism group of an Alexander quandle. In [2], Ferman, Nowik and Teicher also dealt with Alexander quandles of prime order and their (quandle) automorphisms.

In this paper, we will focus on the structure of Alexander quandles from the viewpoint of properties of the group automorphisms. An Alexander quandle M_t is an abelian group M with the binary operation $a *_t b = ta+(1-t)b$ where t is a group automorphism of M. The quandle structure of M_t is completely determined by the group automorphism t. In particular, we will focus on the relationship between M_t and M_{1-t} when both t and $1-t$ are group automorphisms of M. We will show that an Alexander quandle M_t is commutative if and only if $2t = 1$ where 1 is the identity map of M. We will classify commutative Alexander quandles M_t for a finitely generated abelian group M.

2. Main Results

Definition 2.1. A *quandle* is a set Q equipped with a binary operation $* : Q\times Q \to Q$ satisfying the following three axioms.

(i) For all $a \in Q$, $a * a = a$.
(ii) For all $a, b \in Q$, there exists a unique c in Q such that $c * a = b$.
(iii) For all $a, b, c \in Q$, $(a * b) * c = (a * c) * (b * c)$.

For any set Q, a binary operation $* : Q \times Q \to Q$ is defined by $a * b = a$ for all $a, b \in Q$. Then a pair $(Q, *)$ is a quandle, which is called the *trivial quandle*.

Let M be an abelian group and t a group automorphism of M. Define a binary operation $*_t : M \times M \to M$ by

$$a *_t b = ta + (1 - t)b$$

for all $a, b \in M$. It is easy to check that a pair $(M, *_t)$ is a quandle. We call it an *Alexander quandle* and for the sake of simplicity, denote $(M, *_t)$ by M_t.

From the second condition and the third condition in the definition of a quandle, for each $a \in Q$, the function $*a : Q \to Q$, defined by $(*a)(x) = x * a$ for all $x \in Q$, is a quandle automorphism of Q. Therefore, we can define a new operation $\bar{*} : Q \times Q \to Q$ by $a \mathbin{\bar{*}} b = c$ whenever $c * a = b$ for all $a, b \in Q$. It is also a quandle operation on Q, which is called the *r*everse operation of $*$, see [10] for the detail.

Consider the group $\mathrm{Aut}_Q(Q)$ of quandle automorphisms of Q and the free group $F(Q)$ on Q. Define a function $\phi : Q \to \mathrm{Aut}_Q(Q)$ by $\phi(a) = *a$ for all $a \in Q$. Then we obtain a homomorphism from $F(Q)$ to $\mathrm{Aut}_Q(Q)$, indeed, a right action of $F(Q)$ on Q is as an action of quandle automorphisms of Q. A quandle $(Q, *)$ is said to be *connected* if $F(Q)$ acts transitively on Q, that is, for every $x, y \in Q$, there exists $z \in F(Q)$ such that $x * z = y$, see [2].

Definition 2.2. A quandle $(Q, *)$ is said to be *commutative* if $a * b = b * a$ for all $a, b \in Q$.

Remark 2.3. (1) In [8], Joyce introduced an *abelian quandle*, which is a quandle Q satisfying $(a*b)*(c*d) = (a*c)*(b*d)$ for all $a, b, c, d \in Q$, and in [15], Neumann defined a *commutative quandle* Q which is defined by $(a*b)*c = (a*c)*b$ for all $a, b, c \in Q$. The definition of Neumann's commutativity came from the view point of the action of $\mathrm{Aut}_Q(Q)$ on Q. Our definition for commutativity of a quandle is different to both of them.

(2) In [7], Ishii, Iwakiri, Jang and Oshiro introduced the notion of a G-family of quandles, which is motivated by handlebody-knots. One can check that the family $\{M_t\}_{t\in\mathrm{Aut}(M)}$ of Alexander quandles is a G-family of quandles where $G = \mathrm{Aut}(M)$.

Let t be a group automorphism of an abelian group M so that it induces the Alexander quandle M_t. Since the inverse map t^{-1} of t is also a group automorphism of M, one can obtain an Alexander quandle $M_{t^{-1}}$ on M. Note that the quandle operation $*_{t^{-1}}$ is the reverse operation of $*_t$, and that t is the identity map of M if and only if M_t is the trivial quandle.

Since the Alexander quandle structure of M_t is completely determined by the group automorphism t, one can expect a kind of relationship between M_t and M_s when t and s have some relationship as group automorphisms. The following example can give some motivation for this approach.

Example 2.4. Let $M = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ be the cyclic group of order 5. Note that $\mathrm{Aut}(\mathbb{Z}_5)$ consists of four automorphisms t_1, t_2, t_3, t_4, where $t_i : \mathbb{Z}_5 \to \mathbb{Z}_5$ is the group automorphism of $\mathbb{Z}_5$ defined by $t_i(1) = i$. The operation tables of Alexander quandles M_{t_i} are given in Table 1.

Notice that M_{t_1} is the trivial quandle and M_{t_3} is commutative, while the operation table of M_{t_4} is the transpose of the operation table of M_{t_2} as matrices, and that the automorphisms t_2 and t_3 satisfy the relation $t_3 = t_2^{-1}$, but there are no relationship between their quandle structure. Also note that M_{t_3} is commutative, while M_{t_2} is not commutative, and that $a *_{t_2} b = b *_{t_4} a$ for all $a, b \in M$, which is equivalent to the relation $t_4 = 1 - t_2$.

For an abelian group M and $t \in \mathrm{Aut}(M)$, if $1 - t$ is a group automorphism of M, then $a *_t b = b *_{1-t} a$ for all $a, b \in \mathrm{Aut}(M)$. Furthermore if M is finite, then the operation table of M_{1-t} is the transpose of the operation table of M_t as matrices.

Table 1. The operation tables of Alexander quandles of $\mathbb{Z}_5$.

M_{t_1}

$*_{t_1}$	0	1	2	3	4
0	0	0	0	0	0
1	1	1	1	1	1
2	2	2	2	2	2
3	3	3	3	3	3
4	4	4	4	4	4

M_{t_2}

$*_{t_2}$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

M_{t_3}

$*_{t_3}$	0	1	2	3	4
0	0	3	1	4	2
1	3	1	4	2	0
2	1	4	2	0	3
3	4	2	0	3	1
4	2	0	3	1	4

M_{t_4}

$*_{t_4}$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

From the fact that $t_3 = 1 - t_3$, one can see the operation table of M_{t_3} is symmetric, that is, M_{t_3} is commutative.

In general, $1-t$ is not a group automorphism of M. The following theorem gives necessary and sufficient conditions that $1-t$ is a group automorphism of M.

Theorem 2.5. *Let M be a finite abelian group and $t \in \mathrm{Aut}(M)$. The following are equivalent.*

(1) *$1-t$ is a group automorphism of M.*
(2) *t has no fixed points except* 0.
(3) *M_t is connected.*

Proof. Since $1-t$ is a group homomorphism of M, it is sufficient to check that $1-t$ is bijective. Since M is finite, $1-t$ is bijective if and only if $\ker(1-t) = 0$. Since $(1-t)x = 0$ is equivalent to $tx = x$, $\ker(1-t) = 0$ if and only if t has no fixed points except 0. Therefore (1) and (2) are equivalent.

To show that (1) implies (3), let $1-t$ be a group automorphism of M. Then $(1-t)M = M$. For all $x, y \in M$, $y - tx \in M$. Since $(1-t)M = M$, there exists $z \in M$ such that $y - tx = (1-t)z$. Then $x *_t z = tx + (1-t)z = y$ for all $x, y \in M$. Hence M_t is connected.

Conversely, assume that M_t is connected. Since $1-t$ is a group homomorphism of M and M is finite, it is sufficient to show that $1-t$ is surjective. Since M is connected, for each $y \in M$ there exists $z \in F(M)$ such that $0 * z = y$. Put $z = w_1^{e_1} w_2^{e_2} \cdots w_k^{e_k}$ where $w_i \in M$ and $e_i \in \{1, -1\}$ for all $i = 1, 2, \ldots, k$. Since $0 * z = (\cdots((0 *_t w_1^{e_1}) *_t w_2^{e_2}) \cdots) *_t w_k^{e_k} = y$, $t^{e_1+\cdots+e_k}0 + t^{e_2+\cdots+e_k}(1-t^{e_1})w_1 + t^{e_3+\cdots+e_k}(1-t^{e_2})w_2 + \cdots + (1-t^{e_k})w_k = y$. If $e_i = 1$, then $1 - t^{e_i} = 1 - t$. If $e_i = -1$, then $1 - t^{e_i} = t^{-1}t(1-t^{-1}) = -t^{-1}(1-t)$. Therefore $(1-t)m = y$ for some $m \in M$. Hence $1-t$ is surjective. □

Remark 2.6. In [3], Graña showed that M_t is indecomposable if and only if $1-t$ is surjective, and M_t is faithful if and only if $1-t$ is injective. These are equivalent to the condition that $1-t$ is a group automorphism for the finite quandle M.

Example 2.7. Let $\mathbb{Z}_3 \times \mathbb{Z}_3$ be an abelian group of order 9. Define $t : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{Z}_3 \times \mathbb{Z}_3$ by $t(1,0) = (1,1)$ and $t(0,1) = (1,0)$. It is easy to check that t is a group automorphism of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and that t has no fixed points except $(0,0)$. By Theorem 2.5, $1-t$ is also a group automorphism of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The operation tables of M_t and M_{1-t} of $M = \mathbb{Z}_3 \times \mathbb{Z}_3$ are given in Table 2.

The following is a necessary and sufficient condition for an Alexander quandle to be commutative, which is motivated by M_{t_3} in Example 2.4.

Theorem 2.8. *Let M be an abelian group and $t \in \mathrm{Aut}(M)$. Then an Alexander quandle M_t is commutative if and only if $2t = 1$ where 1 is the identity map of M.*

Table 2. The operation tables of Alexander quandles M_t and M_{1-t} of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

M_t

$*_t$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$(0,0)$	$(0,0)$	$(2,1)$	$(1,2)$	$(0,2)$	$(2,0)$	$(1,1)$	$(0,1)$	$(2,2)$	$(1,0)$
$(0,1)$	$(1,0)$	$(0,1)$	$(2,2)$	$(1,2)$	$(0,0)$	$(2,1)$	$(1,1)$	$(0,2)$	$(2,0)$
$(0,2)$	$(2,0)$	$(1,1)$	$(0,2)$	$(2,2)$	$(1,0)$	$(0,1)$	$(2,1)$	$(1,2)$	$(0,0)$
$(1,0)$	$(1,1)$	$(0,2)$	$(2,0)$	$(1,0)$	$(0,1)$	$(2,2)$	$(1,2)$	$(0,0)$	$(2,1)$
$(1,1)$	$(2,1)$	$(1,2)$	$(0,0)$	$(2,0)$	$(1,1)$	$(0,2)$	$(2,2)$	$(1,0)$	$(0,1)$
$(1,2)$	$(0,1)$	$(2,2)$	$(1,0)$	$(0,0)$	$(2,1)$	$(1,2)$	$(0,2)$	$(2,0)$	$(1,1)$
$(2,0)$	$(2,2)$	$(1,0)$	$(0,1)$	$(2,1)$	$(1,2)$	$(0,0)$	$(2,0)$	$(1,1)$	$(0,2)$
$(2,1)$	$(0,2)$	$(2,0)$	$(1,1)$	$(0,1)$	$(2,2)$	$(1,0)$	$(0,0)$	$(2,1)$	$(1,2)$
$(2,2)$	$(1,2)$	$(0,0)$	$(2,1)$	$(1,1)$	$(0,2)$	$(2,0)$	$(1,0)$	$(0,1)$	$(2,2)$

M_{1-t}

$*_{1-t}$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$(0,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(1,1)$	$(2,1)$	$(0,1)$	$(2,2)$	$(0,2)$	$(1,2)$
$(0,1)$	$(2,1)$	$(0,1)$	$(1,1)$	$(0,2)$	$(1,2)$	$(2,2)$	$(1,0)$	$(2,0)$	$(0,0)$
$(0,2)$	$(1,2)$	$(2,2)$	$(0,2)$	$(2,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,1)$
$(1,0)$	$(0,2)$	$(1,2)$	$(2,2)$	$(1,0)$	$(2,0)$	$(0,0)$	$(2,1)$	$(0,1)$	$(1,1)$
$(1,1)$	$(2,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$	$(0,2)$
$(1,2)$	$(1,1)$	$(2,1)$	$(0,1)$	$(2,2)$	$(0,2)$	$(1,2)$	$(0,0)$	$(1,0)$	$(2,0)$
$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(1,2)$	$(2,2)$	$(0,2)$	$(2,0)$	$(0,0)$	$(1,0)$
$(2,1)$	$(2,2)$	$(0,2)$	$(1,2)$	$(0,0)$	$(1,0)$	$(2,0)$	$(1,1)$	$(2,1)$	$(0,1)$
$(2,2)$	$(1,0)$	$(2,0)$	$(0,0)$	$(2,1)$	$(0,1)$	$(1,1)$	$(0,2)$	$(1,2)$	$(2,2)$

Proof. Assume that an Alexander quandle M_t is commutative, that is, $a *_t b = b *_t a$ for all $a, b \in M$. Since $a *_t 0 = 0 *_t a$ for all $a \in M$, $ta = (1-t)a$ and hence $(2t)a = a = 1(a)$ for all $a \in M$ where 1 is the identity map of M. Therefore $2t = 1$. Conversely, let $t \in \mathrm{Aut}(M)$ satisfying $2t = 1$. Since $t = 1 - t$, $a *_t b = b *_t a$ for all $a, b \in M$. Hence the Alexander quandle M_t is commutative. □

By the virtue of the above theorem, we need to know whether there is a group automorphism t of an abelian group M satisfying $2t = 1$. First, we recall the well-known classification theorem of finitely generated abelian groups, see [6].

Proposition 2.9. *Let G be a finitely generated abelian group. Either G is free abelian or there is a list of positive integers $p_1^{s_1}, p_2^{s_2}, \ldots, p_k^{s_k}$, which is unique except for the order of its members such that $p_1, p_2, \ldots, p_k$ are primes, $s_1, s_2, \ldots, s_k$ are positive integers and*

$$G \cong \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}} \oplus \mathbb{Z}^m$$

with $\mathrm{rank}(G) = m$.

Let M be a finitely generated abelian group. If $\mathrm{rank}(M) \geq 1$, then $M \cong \mathbb{Z} \oplus N$ for some abelian group N. Let t be any group automorphism of $\mathbb{Z} \oplus N$. Then $t(1,0) = (a,b)$ for some $(a,b) \in \mathbb{Z} \oplus N$ where 1 is the generator of $\mathbb{Z}$. Since $2t(1,0) = 2(a,b) = (2a, 2b)$ and there is no the inverse element of 2 in $\mathbb{Z}$, $2a \neq 1$ and hence $2t(1,0) = (2a,2b) \neq (1,0)$. Therefore $2t \neq 1$ where 1 is the identity map of $\mathbb{Z} \oplus N$.

It rank$(M) = 0$, then there exist primes $p_1, p_2, \ldots, p_k$ and positive integers $s_1, s_2, \ldots, s_k$ such that $M \cong \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. Assume that $p_i = 2$ for some $i \in \{1, 2, \ldots, k\}$, say $p_1 = 2$. Let t be any group automorphism of $\mathbb{Z}_{2^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. Since $t(1, 0, \ldots, 0) = (a_1, a_2, \ldots, a_k)$ for some $(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_{2^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$ where 1 is the generator of $\mathbb{Z}_{2^{s_1}}$, $2t(1, 0, \ldots, 0) = (2a_1, 2a_2, \ldots, 2a_k)$. Since $gcd(2^{s_1}, 2) \neq 1$, there is no the inverse element of 2 in $\mathbb{Z}_{2^{s_1}}$. Since $2t(1, 0, \ldots, 0) = (2a_1, 2a_2, \ldots, 2a_k) \neq (1, 0, \ldots, 0)$, t does not satisfy the condition $2t = 1$ where 1 is the identity map of $\mathbb{Z}_{2^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$.

Assume that p_i is odd for all $i \in \{1, 2, \ldots, k\}$. Define $t : \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}} \to \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$ by

$$t(x_1, x_2, \ldots, x_k) = \left(\frac{p_1^{s_1} + 1}{2} x_1, \frac{p_2^{s_2} + 1}{2} x_2, \ldots, \frac{p_k^{s_k} + 1}{2} x_k\right)$$

for all $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. For every $i \in \{1, 2, \ldots, k\}$, since p_i is odd, $\frac{p_i^{s_i}+1}{2} \in \mathbb{Z}_{p_i^{s_i}}$. Then t is a well-defined group automorphism. It is easy to prove that t is a group automorphism. Since $2(\frac{p_1^{s_1}+1}{2} x_1, \frac{p_2^{s_2}+1}{2} x_2, \ldots, \frac{p_k^{s_k}+1}{2} x_k) = ((p_1^{s_1}+1)x_1, (p_2^{s_2}+1)x_2, \ldots, (p_k^{s_k}+1)x_k) = (x_1, x_2, \ldots, x_k)$, t satisfies $2t = 1$, where 1 is the identity map of $\mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. Hence M_t is commutative. Thus, one can obtain the following theorem.

Theorem 2.10. *Let M be a finitely generated abelian group.*

(1) *If* rank$(M) \geq 1$, *then M_t is not commutative for any $t \in \mathrm{Aut}(M)$.*
(2) *If* rank$(M) = 0$ *and if $|M|$ is even, then M_t is not commutative for any $t \in$* Aut(M).
(3) *If* rank$(M) = 0$ *and if $|M|$ is odd, then there exists a unique group automorphism t of M such that M_t is commutative.*

Proof. It suffices to show the uniqueness of t in (3). Let t be a group automorphism of $\mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$ satisfying $2t = 1$. For each $i \in \{1, 2, \ldots, k\}$, let e_i be a generator $(0, \ldots, 1, \ldots, 0)$ of $\mathbb{Z}_{p_1^{s_1}} \oplus \cdots \oplus \mathbb{Z}_{p_i^{s_i}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$ of which every entry is zero except the ith entry. Since every abelian group is a $\mathbb{Z}$-module, $t(x_1, x_2, \ldots, x_k) = x_1 t(e_1) + x_2 t(e_2) + \cdots + x_k t(e_k)$ for every $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. Therefore, it is sufficient to determine the image of generators $e_1, e_2, \ldots, e_k$ of $\mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$. For the generator e_1, there exists $(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_{p_1^{s_1}} \oplus \mathbb{Z}_{p_2^{s_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{s_k}}$ such that $t(e_1) = (a_1, a_2, \ldots, a_k)$ and $a_i \in \{0, 1, \ldots, p_i^{s_i} - 1\}$ for each $i \in \{1, 2, \ldots, k\}$. Since $2t = 1$, we obtain $2(a_1, a_2, \ldots, a_k) = (1, 0, \ldots, 0)$, that is, $2a_1 = 1$ in $\mathbb{Z}_{p_1^{s_1}}$ and $2a_i = 0$ in $\mathbb{Z}_{p_i^{s_i}}$ for all $i \in \{2, 3, \ldots, k\}$. Then $2a_1 - 1 = np_1^{s_1}$ for some $n \in \mathbb{Z}$ and $a_i = 0$ for all $i \in \{2, 3, \ldots, k\}$. Since $a_1 = \frac{np_1^{s_1}+1}{2} \in \{0, 1, \ldots, p_1^{s_1} - 1\}$, $0 \leq \frac{np_i^{s_i}+1}{2} \leq p_i^{s_i} - 1$ and hence $n = 0$ or 1. If $n = 0$, then $a_1 = \frac{1}{2}$. It contradicts that a_1 is an integer. If $n = 1$, then $a_1 = \frac{p_1^{s_1}+1}{2}$. Since

Table 3. The operation table of the commutative Alexander quandle of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

$*_t$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
$(0,0)$	$(0,0)$	$(0,2)$	$(0,1)$	$(2,0)$	$(2,2)$	$(2,1)$	$(1,0)$	$(1,2)$	$(1,1)$
$(0,1)$	$(0,2)$	$(0,1)$	$(0,0)$	$(2,2)$	$(2,1)$	$(2,0)$	$(1,2)$	$(1,1)$	$(1,0)$
$(0,2)$	$(0,1)$	$(0,0)$	$(0,2)$	$(2,1)$	$(2,0)$	$(2,2)$	$(1,1)$	$(1,0)$	$(1,2)$
$(1,0)$	$(2,0)$	$(2,2)$	$(2,1)$	$(1,0)$	$(1,2)$	$(1,1)$	$(0,0)$	$(0,2)$	$(0,1)$
$(1,1)$	$(2,2)$	$(2,1)$	$(2,0)$	$(1,2)$	$(1,1)$	$(1,0)$	$(0,2)$	$(0,1)$	$(0,0)$
$(1,2)$	$(2,1)$	$(2,0)$	$(2,2)$	$(1,1)$	$(1,0)$	$(1,2)$	$(0,1)$	$(0,0)$	$(0,2)$
$(2,0)$	$(1,0)$	$(1,2)$	$(1,1)$	$(0,0)$	$(0,2)$	$(0,1)$	$(2,0)$	$(2,2)$	$(2,1)$
$(2,1)$	$(1,2)$	$(1,1)$	$(1,0)$	$(0,2)$	$(0,1)$	$(0,0)$	$(2,2)$	$(2,1)$	$(2,0)$
$(2,2)$	$(1,1)$	$(1,0)$	$(1,2)$	$(0,1)$	$(0,0)$	$(0,2)$	$(2,1)$	$(2,0)$	$(2,2)$

p_1 is odd, $\frac{p_1^{s_1}+1}{2} \in \mathbb{Z}_{p_1^{s_1}}$. Hence $t(1,0,\ldots,0) = (\frac{p_1^{s_1}+1}{2}, 0, \ldots, 0)$. By repeating the same process, we can show that $t(0,\ldots,1,\ldots,0) = (0,\ldots,\frac{p_i^{s_i}+1}{2},\ldots,0)$ for each $i \in \{1,2,\ldots,k\}$. Hence there is one and only one group automorphism t satisfying $2t = 1$. □

Example 2.11. (1) Let $\mathbb{Z}$ be the set of integers. Since $\mathrm{rank}(\mathbb{Z}) = 1$, $(\mathbb{Z}, *_t)$ are not commutative for any $t \in \mathrm{Aut}(\mathbb{Z})$.

(2) Let $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ be an abelian group of order 6. Since $\mathrm{rank}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) = 0$ and $|\mathbb{Z}_2 \oplus \mathbb{Z}_3|$ is even, $(\mathbb{Z}_2 \oplus \mathbb{Z}_3, *_t)$ is not commutative for any $t \in \mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$.

(3) Let $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ be an abelian group of order 9. Since $\mathrm{rank}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) = 0$ and $|\mathbb{Z}_3 \oplus \mathbb{Z}_3|$ is odd, there exists the unique group automorphism t of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ such that $(\mathbb{Z}_3 \oplus \mathbb{Z}_3, *_t)$ is commutative. In fact, the group automorphism t is defined by $t(x_1, x_2) = (2x_1, 2x_2)$ for all $(x_1, x_2) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$. The operation table of an Alexander quandle $(\mathbb{Z}_3 \oplus \mathbb{Z}_3, *_t)$ is given in Table 3.

Remark 2.12. There are commutative Alexander quandles whose underlying abelian group is not finitely generated. For example, consider the product M of countably many copies of $\mathbb{Z}_5$. For the group automorphism t defined by $t(x) = 3x$ for all $x \in M$, M_t is commutative because $2t(x) = 6x = x$.

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References

[1] G. Ehrman, A. Gurpinar, M. Thibault and D. Yetter Toward a classification of finite quandles, *J. Knot Theory Ramifications* **17**(4) (2008) 511–520.

[2] A. Ferman, T. Nowik and M. Teicher, On the structure and automorphism group of finite Alexander quandles, *J. Knot Theory Ramifications* **20**(3) (2011) 463–468.

[3] M. Graña, Indecomposable racks of order p^2, *Beiträge Algebra Geom.* **45** (2004) 665–676.
[4] X. Hou, Automorphism groups of Alexander quandles, *J. Algebra* **344** (2011) 373–385.
[5] X. Hou, Finite module over $\mathbb{Z}[t, t^{-1}]$, *J. Knot Theory Ramifications* **21**(8) (2012) 1250079, 28 pp.
[6] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics (Springer).
[7] A. Ishii, M. Iwakiri, Y. Jang and K. Oshiro, A G-family of quandles and handlebody-knots, preprint (2012), arxiv.org/abs/1205.1855.
[8] D. Joyce, A classifying invariant of knots: The knot quandle, *J. Pure Appl. Algebra* **23** (1982) 37–65.
[9] T. Kepka, Commutative distributive groupoids, *Acta Univ. Carolin. Math. Phys.* **19** (1978) 45–58.
[10] V. O. Manturov, *Knot Theory* (Chapman and Hall/CRC).
[11] S. Matveev, Distributive groupoids in knot theory, *Math. USSR Sb.* **47** (1982) 73–83.
[12] G. Murillo and S. Nelson, Alexander quandles of order 16, *J. Knot Theory Ramifications* **18**(5) (2009) 727.
[13] G. Murillo, S. Nelson and A. Thompsom, Matrices and finite Alexander quandles, *J. Knot Theory Ramifications* **16**(6) (2007) 769–778.
[14] S. Nelson, Classification of finite Alexander quandles, *Topol. Proc.* **27**(1) (2003) 245–258.
[15] B. H. Neumann, *Commutative Quandles*, Lecture Notes in Mathematics, Vol. 1098 (Springer, Berlin, 1984), pp. 81–86.
[16] M. Takasaki, Abstraction of symmetric transformation, *Tohoku Math. J.* **49** (1942) 145–207.